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HOMEOMORPHISMS GROUP OF NORMED VECTOR SPACE: CONJUGACY PROBLEMS AND THE KOOPMAN OPERATOR

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Abstract. This article is concerned with conjugacy problems arising in the homeomorphisms group, Hom(F), of unbounded subsets F of normed vector spaces E. Given two homeomorphisms f and g in Hom(F), it is shown how the existence of a conjugacy may be related to the existence of a common generalized eigenfunction of the associated Koopman operators. This common eigenfunction serves to build a topology on Hom(F), where the conjugacy is obtained as limit of a sequence generated by the conjugacy operator, when this limit exists. The main conjugacy theorem is presented in a class of generalized Lipeomorphisms.

1. **Introduction.** In this article we consider the conjugacy problem in the homeomorphisms group of a finite dimensional normed vector space E. It is out of the scope of the present work to review the problem of conjugacy in general, and the reader may consult for instance [13, 16, 29, 33, 26, 42, 45, 51, 52] and references therein, to get a partial survey of the question from a dynamical point of view. The present work raises the problem of conjugacy in the group Hom(F) consisting of homeomorphisms of an unbounded subset F of E and is intended to demonstrate how the conjugacy problem, in such a case, may be related to spectral properties of the associated Koopman operators. In this sense, this paper provides new insights on the relations between the spectral theory of dynamical systems [5, 17, 23, 36] and the topological conjugacy problem [51, 52][1].

More specifically, given two homeomorphisms f and g of F, we show here that the conjugacy problem in Hom(F) is related to the existence of a common generalized

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[1] Usually, the spectral theory of dynamical systems makes usage of concepts from ergodic theory, but the latter are not required in the description of the relationships that we propose below.

eigenfunction for the associated Koopman operators U_{f} and U_{g} (cf. Definition 2.2), *i.e.* a function Φ satisfying,

$$\begin{cases} U_{\mathsf{f}}(\Phi) \geq \lambda\Phi, \\ U_{\mathsf{g}}(\Phi) \geq \mu\Phi, \end{cases} \tag{1}$$

for some $\lambda, \mu > 0$, where Φ lives within some cone K of the set of continuous real-valued functions on F. The elements of this cone possess the particularity of exhibiting a behavior at infinity prescribed by a subadditive function R; see Section 2.

More precisely, when such a Φ exists, it is shown how Φ can be used to build a topology such that the sequence of iterates $\{\mathcal{L}^{\mathsf{n}}_{\mathsf{f},\mathsf{g}}(h_0)\}_{\mathsf{n}\in\mathbb{N}}$, of the conjugacy operator[2] initiated to some $h_0 \in \mathrm{Hom}(F)$ close enough to $\mathcal{L}_{\mathsf{f},\mathsf{g}}(h_0)$ in that topology, converges to the conjugacy h satisfying $f \circ h = h \circ g$, provided that $\{\mathcal{L}^{\mathsf{n}}_{\mathsf{f},\mathsf{g}}(h_0)\}$ is bounded on every compact subset of F; cf. Theorem 4.3. The topology built from Φ relies on a premetric on $\mathrm{Hom}(F)$ where Φ serves to weigh the distance to the identity of any homeomorphism of F; see Eqs. (3) and (13) below.

The plan of this article is as follows. Section 2 sets up the functional framework used in this article, where in particular the main properties of the topology built from any member $\Phi \in K$ are derived with particular attention to closure properties and convergence in that topology of sequences in $\mathrm{Hom}(F)$; cf Propositions 3 and 4. Section 3 establishes a fixed point theorem, Theorem 3.1, for mappings acting on $\mathrm{Hom}(F)$, when this group is endowed with the topology discussed in Section 2. In section 4 the main theorem of conjugacy, Theorem 4.3, is proved based on Theorem 3.1 applied to the conjugacy operator, where the contraction property is shown to be related to the existence of a common generalized eigenfunction Φ of a generalized eigenvalue problem of type (1). This related generalized eigenvalue problem for the Koopman operators associated with the conjugacy problem is then discussed in Section 4.3 where, in particular, connections with relatively recent results about functional equations such as the Schröder equations and the Abel equation are established. Concluding remarks regarding the possible extensions of the present work are presented in Section 5. The results obtained in the present study were motivated in part by [14], where results are derived for the conjugacy problem on not necessarily compact manifolds. Connections with topological equivalence problems between periodic vector fields and autonomous ones as considered in [14], will be discussed elsewhere.

2. A functional framework on the homeomorphisms group.

In this section we introduce a family of subgroups of homeomorphisms for the composition law. These subgroups associated with the framework from which they are derived, will be used in the analysis of the conjugacy problem in the homeomorphisms group itself. The topology with which they are endowed is introduced here and the main properties are derived. The extension of these topologies to the whole group of homeomorphisms is also presented and the related closure properties and convergence of sequences in the homeomorphisms group are discussed.

2.1. Notations and preliminaries.

In this article E denotes a d-dimensional normed vector space ($d \in \mathbb{N}^*$), endowed with a norm denoted by $\|\cdot\|$ and F denotes an unbounded subset of E. The following class of functions serves to specify some

[2] where for f and g given in $\mathrm{Hom}(F)$, $\mathcal{L}_{f,g} : \psi \mapsto f \circ \psi \circ g^{-1}$ is acting on $\psi \in \mathrm{Hom}(F)$.

behavior at infinity of homeomorphisms and to build topologies that will be central in our approach; cf Proposition 2.

Definition 2.1. The space $\mathcal{E}_{\mathsf{F}}^{\mathsf{R}}$. Let $R:\mathbb{R}^+\to\mathbb{R}^+-\{0\}$, be a continuous function, which is subadditive, *i.e.*,

$$R(u+v)\leq R(u)+R(v),\ \forall\, u,v\in\mathbb{R}^+.$$

We denote by $\mathcal{E}_{\mathsf{F}}^{\mathsf{R}}$ the set of continuous functions $\Phi:F\to\mathbb{R}^+$, satisfying:

(G_1) $\exists\, m>0,\ \forall\, x\in F,\ m\leq\Phi(x)$,
(G_2) Coercivity condition: $\Phi(x)\longrightarrow+\infty$, as $x\in F$ and $\|x\|\to+\infty$,
(G_3) Cone condition: There exist β and γ, such that $\beta>\gamma>0$, and,

$$\forall\, x\in F,\ \gamma R(\|x\|)\leq\Phi(x)\leq\beta R(\|x\|). \tag{2}$$

For obvious reasons, R will be called sometimes a growth function.

Remark 1. (a) It is interesting to note that the closure $K:=\overline{\mathcal{E}_{\mathsf{F}}^{\mathsf{R}}}$, is a closed cone with non-empty interior in the Banach space $X=C^0(F,\mathbb{R})$ of continuous functions $\Psi:F\to\mathbb{R}$, endowed with the compact-open topology [25], *i.e.* $K+K\subset K$, $tK\subset K$ for every $t\geq0$, $K\cap(-K)=\{0_{\mathsf{X}}\}$ and Int $K\neq\emptyset$.

(b) Note that the results obtained in this article could be derived with weaker assumptions than in (G_3), such as relaxing (2) for $\|x\|\geq\nu$ for some $\nu>0$, and assuming measurability on R and Φ (with respect to the Borel σ-algebras of $\mathbb{R}^+$ and F respectively) instead of continuity. However, further properties have to be derived in order to extend appropriately the approach developed in this paper. For instance assuming only measurability of R, it can be proved, since R is assumed to be subadditive, that R is bounded on compact subsets of $\mathbb{R}^+$, *e.g.* [24, lemma 1, p. 167]; a property that would appear to be important for extending the results of this article in such a context. We leave for the interested reader these possible extensions of the results presented hereafter.

(c) Other generalization about R could be also considered, such as $R(u+v)\leq C(R(u)+R(v)),\ \forall\, u,v\in\mathbb{R}^+$, for some $C>0$, allowing for the fact that any positive power of a subadditive function is subadditive in that sense; but this condition would add complications in the proof of Theorem 3.1 for instance. We do not enter in all these generalities to make the expository less technical.

We need also to consider a function $r:\mathbb{R}^+\to\mathbb{R}^+$, verifying the following assumptions.

Assumptions on the function r. We assume that $r(x)=0$ if and only if $x=0$, r is continuous at 0, r is nondecreasing, subadditive and for some statements we will assume furthermore that,

(A_{r}) r is lower semi-continuous for the pointwise convergence on F, *i.e.*,

$$\forall\, x\in F,\ r(\liminf_{\mathsf{n}\to+\infty}\|f_{\mathsf{n}}(x)\|)\leq\liminf_{\mathsf{n}\to+\infty}r(\|f_{\mathsf{n}}(x)\|),$$

for any sequence $\{f_{\mathsf{n}}\}_{\mathsf{n}\in\mathbb{N}}$ of self-mappings of F.

Cross condition. Finally, we will consider the following cross condition between the growth function R and the function r,

($C_{\mathsf{r,R}}$) $\exists\, a>0,\ \exists\, b>0,\ \forall\, u\in\mathbb{R}^+,\ R(u)\leq ar(u)+b.$

As simple example of functions Φ, R, and r satisfying the above conditions (including ($\mathrm{A_r}$)), we can cite $r(u) = u$, $\Phi(x) = R(\|x\|) = \sqrt{\|x\|} + 1$, that will be used to illustrate the main theorem of this article later on; see subsection 4.2.

Hereafter in this subsection, condition ($\mathrm{A_r}$) is not required. We introduce now the following functional on $\mathrm{Hom}(F)$ with possible infinite values,

$$|\cdot|_{\Phi,r} : \begin{cases} \mathrm{Hom}(F) \to \overline{\mathbb{R}^+} \\ f \mapsto |f|_{\Phi,r} := \sup_{x\in F}\Big(\frac{r(\|f(x)-x\|)}{\Phi(x)}\Big). \end{cases} \tag{3}$$

Note that,

$$|f|_{\Phi,r} = 0 \text{ if and only if } f = \mathrm{Id}_F \text{ (separation condition)},$$

where Id_F denotes the identity map of F.

Definition 2.2. The Koopman operator with domain $\mathcal{E}_F^R$, associated with $f \in \mathrm{Hom}(F)$, is defined as the operator U_f given by:

$$U_f : \begin{cases} \mathcal{E}_F^R \to C^0(F, \mathbb{R}^+) \\ \Phi \mapsto U_f(\Phi), \text{ where } U_f(\Phi)(x) = \Phi(f(x)), \ \forall\, x \in F. \end{cases} \tag{4}$$

Remark 2. Classically, the Koopman operator is given with other domain such as $L^p(F)$ [22, 36] (generally $p = 2$) and arises naturally with the Frobenius-Perron operator in the study of ergodicity and mixing properties of measure-preserving transformations; *e.g.* [5, 17, 36]. The Koopman operator addresses the evolution of phase space functions (observables), such as Φ above, described by the linear operator U_f rather than addressing a direct study of the nonlinear dynamics generated by f. This idea has been introduced by Koopman and von Neumann in the early 30's [32], and has paved the road of what is called today the spectral analysis of dynamical systems [5, 17, 23, 36]. We propose here more specifically to link this spectral analysis with the topological problem of conjugacy [51, 52], from an abstract point of view. Let us mention nevertheless, that related relationships are known to exist in certain instances, but a general treatment of the question is missing to the best of the authors' knowledge. For example, it is known that ergodic transformations of a compact manifold are semiconjugate to a rotation on the circle if and only if there exists a non-constant eigenfunction of the Koopman operator associated with the eigenvalue $e^{-2i\pi\omega}$ for some $\omega \in \mathbb{Q}$; *e.g.* [40, Proposition 8]. We consider hereafter conjugacy problems between elements of a single group of transformations ($\mathrm{Hom}(F)$), rather than semiconjugacy problems between elements of different groups [29]. Our phase space will be also always assumed to be unbounded (and thus non-compact).

Remark 3. Note that in general U_f, as defined in Definition 2.2, does not leave stable $\mathcal{E}_F^R$, since $U_f(\Phi)$ is not guaranteed to satisfy (G_3) for $\Phi \in \mathcal{E}_F^R$. In fact a direct analysis shows that in order to have $U_f(\Phi)$ to satisfiy (G_3) it requires restrictions on f and R that we want to avoid[3]. However if there exists some positive constants $c(f)$ and $C(f)$ such that $c(f)\Phi \leq U_f(\Phi) \leq C(f)\Phi$ for a particular Φ in $\mathcal{E}_F^R$, then we can conclude that $U_f(\Phi)$ lives in $\mathcal{E}_F^R$. In the proposition below, we derive when $f \in \mathbb{H}_{\Phi,r}$ such a upper bound valid for all Φ in $\mathcal{E}_F^R$; see (6). The lower bound will

[3] For instance if we assume $c\|x\| \leq \|f(x)\| \leq C\|x\|$ for all $x \in F$, and R to be furthermore increasing and quasi-homogeneous [43], then $U_f(\Phi)$ satisfies (G_3). Assuming furthermore that $\|f(x)\| \to \infty$ as $\|x\| \to \infty$, then $U_f(\Phi)$ satisfies (G_2) and U_f leaves thus stable $\mathcal{E}_F^R$ in such a case.

be naturally verified for generalized eigenfunctions of $U_{\mathbf{f}}$ in the sense of Definition 4.2 below; making such eigenfunctions elements of $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$.

We can now state the following proposition.

Proposition 1. *Consider R given as in Definition 2.1, and $\Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$. Let r satisfy the above assumptions except $(A_{\mathbf{r}})$, and such that $(C_{\mathbf{r},\mathbf{R}})$ is satisfied. Introduce the following subset of* $\mathrm{Hom}(F)$*,*

$$\mathbb{H}_{\Phi,\mathbf{r}} := \{f \in \mathrm{Hom}(F) \;:\; |f|_{\Phi,\mathbf{r}} < \infty, \;\; \text{and } |f^{-1}|_{\Phi,\mathbf{r}} < \infty\}. \tag{5}$$

Then $(\mathbb{H}_{\Phi,\mathbf{r}}, \circ)$ is a subgroup of $(\mathrm{Hom}(F),\circ)$ and, for any $f \in \mathbb{H}_{\Phi,\mathbf{r}}$, the Koopman operator, $U_{\mathbf{f}}$, associated with f is a bounded operator on $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ which satisfies,

$$\forall\, \Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}},\; U_{\mathbf{f}}(\Phi) \leq \Lambda(f)\Phi, \tag{6}$$

with,

$$\Lambda(f) := a\beta|f|_{\Phi,\mathbf{r}} + bm^{-1}\beta + \beta\gamma^{-1} < \infty, \tag{7}$$

and where the constants appearing here are as introduced above.

Proof. We first prove the subgroup property. Let x be arbitrary in F, and f, g in $\mathbb{H}_{\Phi,\mathbf{r}}$. Then,

$$\begin{aligned}\frac{r(\|f\circ g^{-1}(x) - x\|)}{\Phi(x)} \leq \frac{r(\|f\circ g^{-1}(x) - g^{-1}(x)\|)}{\Phi(g^{-1}(x))} \cdot \frac{\Phi(g^{-1}(x))}{\Phi(x)} \\ + \frac{r(\|g^{-1}(x) - x\|)}{\Phi(x)}.\end{aligned} \tag{8}$$

From (G_3) and the subadditivity of R,

$$\Phi(g^{-1}(x)) \leq \beta(R(\|g^{-1}(x) - x\|) + R(\|x\|)),$$

and since $R(\|x\|) \leq \gamma^{-1}\Phi(x)$, we get by using $(C_{\mathbf{r},\mathbf{R}})$ and (G_1),

$$\begin{aligned}\frac{\Phi(g^{-1}(x))}{\Phi(x)} &\leq \beta\left(\frac{a\, r(\|g^{-1}(x) - x)\|)}{\Phi(x)} + \frac{b}{\Phi(x)}\right) + \beta\gamma^{-1} \\ &\leq C := a\beta|g^{-1}|_{\Phi,\mathbf{r}} + bm^{-1}\beta + \beta\gamma^{-1},\end{aligned} \tag{9}$$

with C finite since $|g^{-1}|_{\Phi,\mathbf{r}}$ exists by definition of $\mathbb{H}_{\Phi,\mathbf{r}}$.

Going back to (8) we deduce that,

$$\frac{r(\|f\circ g^{-1}(x) - x\|)}{\Phi(x)} \leq C|f|_{\Phi,\mathbf{r}} + |g^{-1}|_{\Phi,\mathbf{r}} < \infty, \tag{10}$$

which concludes that $f\circ g^{-1} \in \mathbb{H}_{\Phi,\mathbf{r}}$, and $\mathbb{H}_{\Phi,\mathbf{r}}$ is a subgroup of $\mathrm{Hom}(F)$. The proof of (6) consists then just in a reinterpretation of (9). ☐

Remark 4. Fairly general homeomorphisms are encompassed by the groups, $\mathbb{H}_{\Phi,\mathbf{r}}$, introduced above. For instance, in the special case $\Phi(x) = R(\|x\|) := \|x\| + 1$, and $r(x) = x$, denoting by $\mathbb{H}_0$ the group $\mathbb{H}_{\Phi,\mathbf{r}}$, and $|\cdot|_{\Phi,\mathbf{r}}$ by $|\cdot|_0$ for that particular choice of Φ, and r, the following two classes of homeomorphisms belong to $\mathbb{H}_0$ and exhibit non-trivial dynamics.

(a) Mapping f of $\mathbb{R}^{\mathbf{d}}$ which are perturbation of linear mapping in the following sense:

$$f(x) = Tx + \varphi(x), \tag{11}$$

with T a *linear automorphism* of $\mathbb{R}^{\mathbf{d}}$ and φ a C^1 map which is globally Lipschitz with Lipschitz constant, $\mathrm{Lip}(\varphi)$, satisfying $\mathrm{Lip}(\varphi) < \|T^{-1}\|^{-1}_{\mathcal{L}(\mathbb{R}^d)}$ — that ensures f to be an homeomorphism of $\mathbb{R}^{\mathsf{d}}$ from the *Lipschitz inverse mapping theorem* (cf. *e.g.* [26, p. 244]) — and $\|\varphi(x)\| \leq C(\|x\|+1)$ (that ensures $|f|_0 < \infty$) for some positive constant; and such that the inverse of the differential of f has a uniform upper bound $M > 0$ in the operator norm, *i.e.* $\|[Df(u)]^{-1}\|_{\mathcal{L}(\mathbb{R}^d)} \leq M$ for every $u \in \mathbb{R}^{\mathsf{d}}$, which ensures $|f^{-1}|_0 < \infty$ by the mean value theorem. For instance, $f(x) = x + \frac{1}{2}\log(1+x^2)$ provides such an homeomorphism of $\mathbb{R}$. Note that every φ that is a C^1 map of $\mathbb{R}^{\mathsf{d}}$ with compact support and with appropriate control on its differential leads to an homeomorphism of type (11) that belongs to $\mathbb{H}_0$.

(b) Extensions of the preceding examples to the class of non-smooth (non C^1) perturbations of linear automorphisms can be considered; exhibiting non-trivial homeomorphisms of $\mathbb{H}_0$. For instance, the two-parameter family of homeomorphisms, $\{\mathbb{L}_{\mathsf{a,b}}\ ,\ b \in \mathbb{R}\backslash\{0\},\ a \in \mathbb{R}\}$, known as the Lozi maps family [39]:

$$\begin{aligned} \mathbb{L}_{\mathsf{a,b}} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\mapsto (1-a|x|+y, bx) \end{aligned}, \tag{12}$$

where $|\cdot|$ denotes the absolute value here; constitutes a family of elements of $\mathbb{H}_0$. Indeed, it is not difficult to show that $\mathbb{L}_{\mathsf{a,b}}$ and its inverse, for $b \neq 0$, $\mathbb{M}_{\mathsf{a,b}} : (u,v) \mapsto (\frac{1}{\mathsf{b}}v, -1+u+\frac{\mathsf{a}}{|\mathsf{b}|}|v|)$ have finite $|\cdot|_0$-values. This family shares similar properties with the Hénon maps family. For instance there exists an open set in the parameter space for which generalized hyperbolic attractors exist [41].

We introduce now the following functional on $\mathbb{H}_{\Phi,\mathsf{r}} \times \mathbb{H}_{\Phi,\mathsf{r}}$,

$$\rho_{\Phi,\mathsf{r}}(f,g) := \max(|f \circ g^{-1}|_{\Phi,\mathsf{r}}, |f^{-1} \circ g|_{\Phi,\mathsf{r}}), \tag{13}$$

which is well-defined by Proposition 1 and non-symmetric. Since obviously $\rho_{\Phi,\mathsf{r}}(f,g) \geq 0$ whatever f, and g, and $\rho_{\Phi,\mathsf{r}}(f,g) = 0$ if and only if $f = g$, then $\rho_{\Phi,\mathsf{r}}$ is in fact a premetric on $\mathbb{H}_{\Phi,\mathsf{r}}$. Note that hereafter, we will simply denotes $f \circ g$ by fg. Due to the non-symmetric property, two natural types of "balls" can be defined with respect to the premetric $\rho_{\Phi,\mathsf{r}}$. More precisely:

Definition 2.3. An open $\rho_{\Phi,\mathsf{r}}$-ball of center f to the right (resp. left) and radius $\alpha > 0$ is the subset of $\mathbb{H}_{\Phi,\mathsf{r}}$ defined by $B^+_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha) := \{g \in \mathbb{H}_{\Phi,\mathsf{r}}\ :\ \rho_{\Phi,\mathsf{r}}(g,f) < \alpha\}$ (resp. $B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha) := \{g \in \mathbb{H}_{\Phi,\mathsf{r}}\ :\ \rho_{\Phi,\mathsf{r}}(f,g) < \alpha\}$).

Proposition 2. *Consider R given as in Definition 2.1, and $\Phi \in \mathcal{E}^{\mathsf{R}}_{\mathsf{F}}$. Let r satisfy the above assumptions except (A_{r}), and such that ($C_{\mathsf{r,R}}$) is satisfied. Then, the premetric as defined in (13) satisfies the following properties.*

(i) *For every f, g, h, in $\mathbb{H}_{\Phi,\mathsf{r}}$, the following relaxed triangle inequality holds,*

$$\rho_{\Phi,\mathsf{r}}(f,g) \leq a\beta\rho_{\Phi,\mathsf{r}}(f,h)\rho_{\Phi,\mathsf{r}}(h,g) + (bm^{-1}\beta + \beta\gamma^{-1})\rho_{\Phi,\mathsf{r}}(f,h) + \rho_{\Phi,\mathsf{r}}(h,g). \tag{14}$$

(ii) *The following families of subsets of $\mathbb{H}_{\Phi,\mathsf{r}}$,*

$$\mathfrak{T}^+(\rho_{\Phi,\mathsf{r}}) := \{\mathcal{H} \subset \mathbb{H}_{\Phi,\mathsf{r}}\ :\ \forall\ f \in \mathcal{H}, \exists\ \alpha > 0,\ B^+_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha) \subset \mathcal{H}\}$$

and,

$$\mathfrak{T}^-(\rho_{\Phi,\mathsf{r}}) := \{\mathcal{H} \subset \mathbb{H}_{\Phi,\mathsf{r}}\ :\ \forall\ f \in \mathcal{H}, \exists\ \alpha > 0,\ B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha) \subset \mathcal{H}\}$$

are two topologies on $\mathbb{H}_{\Phi,\mathsf{r}}$.

(iii) *For all* $f \in \mathbb{H}_{\Phi,\mathbf{r}}$*, for all* $\alpha^* > 0$*, and for all* $g \in B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha^*)$*, the following property holds:*

$$\Big(\rho_{\Phi,\mathbf{r}}(f,g) < \frac{\alpha^*}{bm^{-1}\beta + \beta\gamma^{-1}})\Big) \Rightarrow (\exists\ \alpha > 0,\ B^-_{\boldsymbol{\rho}_{\Phi,r}}(g,\alpha) \subset B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha^*)),$$

and thus for all $f \in \mathbb{H}_{\Phi,\mathbf{r}}$*,* $\bigcup_{\boldsymbol{\alpha}>0} B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha)$ *is a fundamental system of neighborhoods of* f*, which renders* $\mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$ *first-countable. An analogous statement holds with "+" instead of "-".*

(iv) *Let* $\overline{\mathbb{H}}^{(-)}_{\Phi,\mathbf{r}}$ *denote the closure of* $\mathbb{H}_{\Phi,\mathbf{r}}$ *for the topology* $\mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$*, then*

$$\overline{\mathbb{H}}^{(-)}_{\Phi,\mathbf{r}} \bigcap \mathrm{Hom}(F) \subset \mathbb{H}_{\Phi,\mathbf{r}}.$$

Remark 5. Proof of (iii) below shows that an arbitrary open $\rho_{\Phi,\mathbf{r}}-$ball (centered to the right or left) is not necessarily open in the sense of not being an element $\mathcal{H}$ of $\mathfrak{T}^{+/-}(\rho_{\Phi,\mathbf{r}})$, since $b > 0$ and $\gamma < \beta$.

Proof. We first prove (i). Using the triangle inequality for $\|\cdot\|$ and subadditivity of r, it is easy to note that for all $x \in F$, and all $f, g, h \in \mathbb{H}_{\Phi,\mathbf{r}}$,

$$\frac{r(\|fg^{-1}(x) - x\|)}{\Phi(x)} \leq \frac{r(\|fh^{-1}hg^{-1}(x) - hg^{-1}(x)\|)}{\Phi(x)} + |hg^{-1}|_{\Phi,\mathbf{r}}. \tag{15}$$

From the following trivial equality,

$$r(\|fh^{-1}hg^{-1}(x) - hg^{-1}(x)\|) = r(\|fh^{-1}hg^{-1}(x) - hg^{-1}(x)\|)\frac{\Phi(hg^{-1}(x))}{\Phi(hg^{-1}(x))},$$

we deduce from Proposition 1, that,

$$\frac{r(\|fg^{-1}(x) - x\|)}{\Phi(x)} \leq \Lambda(hg^{-1})|fh^{-1}|_{\Phi,\mathbf{r}},$$

where $\Lambda(hg^{-1})$ is well-defined since $\mathbb{H}_{\Phi,\mathbf{r}}$ is a subgroup. This last inequality reported in (15) gives then,

$$\sup_{\mathbf{x}\in\mathbf{F}}\Big(\frac{r(\|fg^{-1}(x) - x\|)}{\Phi(x)}\Big) \leq \Lambda(hg^{-1})|fh^{-1}|_{\Phi,\mathbf{r}} + |hg^{-1}|_{\Phi,\mathbf{r}}, \tag{16}$$

leading to (14), by re-writing appropriately (16) and repeating the computations with the substitutions $f \leftarrow f^{-1}$, $g^{-1} \leftarrow g$ and $h^{-1} \leftarrow h$ for the estimation of $|f^{-1}g|_{\Phi,\mathbf{r}}$.

The proof of (ii) is just a classical "game" with the axioms of a topology and is left to the reader.

We prove now (iii), only for $\mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$; the proof for $\mathfrak{T}^+(\rho_{\Phi,\mathbf{r}})$ being a repetition. Let $f \in \mathbb{H}_{\Phi,\mathbf{r}}$ and $\alpha^* > 0$. Let $g \in B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha^*)$, then from (14) we get for all $h \in \mathbb{H}_{\Phi,\mathbf{r}}$,

$$\rho_{\Phi,\mathbf{r}}(f,h) \leq a\beta\rho_{\Phi,\mathbf{r}}(f,g)\rho_{\Phi,\mathbf{r}}(g,h) + (bm^{-1}\beta + \beta\gamma^{-1})\rho_{\Phi,\mathbf{r}}(f,g) + \rho_{\Phi,\mathbf{r}}(g,h). \tag{17}$$

We seek now the existence of $\alpha > 0$ such that $B^-_{\boldsymbol{\rho}_{\Phi,r}}(g,\alpha) \subset B^-_{\boldsymbol{\rho}_{\Phi,r}}(f,\alpha^*)$. Denoting $\rho_{\Phi,\mathbf{r}}(f,g)$ by α', such a problem of existence is then reduced from Eq. (17) to the existence of a solution $\alpha > 0$ of,

$$a\beta\alpha\alpha' + (bm^{-1}\beta + \beta\gamma^{-1})\alpha' + \alpha < \alpha^*. \tag{18}$$

A necessary condition of existence is,

$$\alpha' < \alpha^{**} := \frac{\alpha^*}{bm^{-1}\beta + \beta\gamma^{-1}} \tag{19}$$

that turns out to be sufficient since any $\alpha > 0$ satisfying,

$$\alpha < \frac{\alpha^* - (bm^{-1}\beta + \beta\gamma^{-1})\alpha'}{1 + a\beta\alpha'}, \tag{20}$$

is a solution because the RHS is positive.

The second part of (iii) is a reinterpretation of the result just obtained. Indeed, we have proved that for all $f \in \mathbb{H}_{\Phi,\mathbf{r}}$, and for all $\alpha^* > 0$ there exists $0 < \alpha^{**} < \alpha^*$ (since $\gamma < \beta$ and $b > 0$ by definition), such that $B^-_{\boldsymbol{\rho}_{\Phi,r}}(f, \alpha^{**}) \in \mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$.

Now if we introduce $\mathbb{B}(f) := \bigcup_{\alpha^* > 0} B^-_{\boldsymbol{\rho}_{\Phi,r}}(f, \alpha^*)$, then the family $\mathcal{F}(f)$ of subsets of $\mathbb{H}_{\Phi,\mathbf{r}}$ defined by,

$$\mathcal{F}(f) := \{V \in 2^{\mathbb{H}_{\Phi,r}} \ : \ \exists\, B \in \mathbb{B}(f), \text{ s. t. } V \supset B\},$$

is a family of neighborhoods of f, since every $V \in \mathcal{F}(f)$ contains by definition a subset $B^-_{\boldsymbol{\rho}_{\Phi,r}}(f, \alpha^*)$ (that is not necessarily open), and therefore a subset of type $B^-_{\boldsymbol{\rho}_{\Phi,r}}(f, \alpha^{**})$ which is open from what precedes. Thus first countability naturally holds and the proof of (iii) is complete.

We prove now (iv). Let $f \in \overline{\mathbb{H}}^{(-)}_{\Phi,\mathbf{r}} \bigcap \mathrm{Hom}(F)$, then by the property (iii), there exists a sequence $\{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}} \in \mathbb{H}^{\mathbb{N}}_{\Phi,\mathbf{r}}$, such that $\rho_{\Phi,\mathbf{r}}(f, f_{\mathbf{n}}) \underset{\mathbf{n}\to\infty}{\to} 0$. Then by definition of $\rho_{\Phi,\mathbf{r}}$, we get in particular that $|f f_{\mathbf{n}}^{-1}|_{\Phi,\mathbf{r}}$ and $|f^{-1} f_{\mathbf{n}}|_{\Phi,\mathbf{r}}$ exist from which we deduce that $|f|_{\Phi,\mathbf{r}}$ and $|f^{-1}|_{\Phi,\mathbf{r}}$ exist, since $\mathbb{H}_{\Phi,\mathbf{r}}$ is a group. □

2.2. **Closure properties for extended premetric on Hom(F).** In this section, we extend the closure property (iv) of Proposition 2 to Hom(F) itself (cf. Proposition 3) and prove a cornerstone proposition (Proposition 4) concerning the convergence in $\mathfrak{T}^-(\rho'_{\Phi,\mathbf{r}})$ of sequences taking values in Hom(F), where $\rho'_{\Phi,\mathbf{r}}$ denotes the extension of the premetric $\rho_{\Phi,\mathbf{r}}$ to Hom(F)×Hom(F).

The result described in Proposition 3 will allow us to make precise conditions for which the solution of the fixed point theorem proved in the next section, lives in $\mathbb{H}_{\Phi,\mathbf{r}}$. This specific result is not fundamental for the proof of the main theorem of conjugacy of this article, Theorem 4.3; whereas Proposition 4 will play an essential role in the proof of the fixed point theorem, Theorem 3.1, and by the way in the proof of Theorem 4.3. Important related concepts such as the one of incrementally bounded sequence are also introduced in this subsection.

We define $\rho'_{\Phi,\mathbf{r}}$ as the extension of the premetric $\rho_{\Phi,\mathbf{r}}$ to Hom(F)×Hom(F), by classically allowing $\rho'_{\Phi,\mathbf{r}}$ to take values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ instead of $\mathbb{R}$; with the usual extensions of the arithmetic operations[4].

According to this basic extension procedure, it can be shown that $\mathfrak{T}^{+/-}(\rho'_{\Phi,\mathbf{r}})$ is a topology on Hom(F) and that Proposition 2 can be reformulated for $\rho'_{\Phi,\mathbf{r}}$ with the appropriate modifications. Note that, in particular, the relaxed triangle inequality (14) holds for $\rho'_{\Phi,\mathbf{r}}$ and any f, g and h in Hom(F). Indeed, either fg^{-1} and $f^{-1}g$ both belong to $\mathbb{H}_{\Phi,\mathbf{r}}$ in which case (14) obviously holds; or fg^{-1} and $f^{-1}g$ do not

[4]Note that similarly, $|\cdot|_{\Phi,r}$ may be extended in such a way, but it is important to have in mind that $|f|'_{\Phi,r} = \infty$ and $|g|'_{\Phi,r} = \infty$ do not necessarily imply that $\rho'_{\Phi,r}(f,g) = \infty$. This is, for instance, the case for $f(x) = g(x) = Ax$, with $A \in Gl_d(\mathbb{R})$, $A \neq I_d$, $r(x) = x$, and $\Phi(x) = \sqrt{\|x\|}+1$.

both belong to $\mathbb{H}_{\Phi,\mathbf{r}}$. In the latter case, because of the group structure of $\mathbb{H}_{\Phi,\mathbf{r}}$, at least one element in $\{fh^{-1}, f^{-1}h, hg^{-1}, h^{-1}g\}$ does not belong to $\mathbb{H}_{\Phi,\mathbf{r}}$. This leads to the conclusion that the inequalities (14) still hold for $\rho'_{\Phi,\mathbf{r}}$.

We are now in a position to introduce contingent conditions to our framework that are required to obtain closure type results in $\mathrm{Hom}(F)$. These conditions possess the particularity to hold in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ for any sequences involved in the closure problem related to the topology $\mathfrak{T}^-(\rho'_{\Phi,\mathbf{r}})$. More precisely we introduce the following concepts.

Definition 2.4. We say that a sequence $\{f_{\mathbf{n}}\}$ of elements of $\mathrm{Hom}(F)$ is incrementally bounded in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ with respect to q, for some $q \in \mathbb{N}$, if and only if:

$$\exists\, C^+_{\mathbf{q}},\ \forall\, p \in \mathbb{N},\ (p \geq q) \Rightarrow (\rho'_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, f_{\mathbf{q}}) \leq C^+_{\mathbf{q}}).$$

When no further conditions on q are assumed, we say that the sequence is incrementally bounded.

Furthermore, the sequence is said to be uniformly bounded in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ if and only if:

$$\exists\, C^+,\ \forall\, p, q \in \mathbb{N},\ (p \geq q) \Rightarrow (\rho'_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, f_{\mathbf{q}}) \leq C^+_{\mathbf{q}}).$$

The "(-)-statements" consist of changing the role of p and q in the above statements. We denote by $\mathcal{IB}^+$ the set of incrementally bounded sequences in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ and by $\mathcal{IB}^+_{\mathbf{u}}$ its subset constituted only by uniformly incrementally bounded sequences.

Definition 2.5. We define $\overline{\mathrm{Hom}(F)}^{(-),\mathbf{b}^+}$ to be the set consisting of all the limit points in $\mathfrak{T}^-(\rho'_{\Phi,\mathbf{r}})$ of sequences $\{f_{\mathbf{n}}\} \in \mathrm{Hom}(F)$, such that,

$$\exists\, n_0 \in \mathbb{N}\ :\ f_{\mathbf{n}_0} \in \mathbb{H}_{\Phi,\mathbf{r}}\ ,$$

for which $\{f_{\mathbf{n}}\}$ is incrementally bounded in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ with respect to n_0.

We have then the following important proposition that completely characterizes the limits in $\mathfrak{T}^-(\rho'_{\Phi,\mathbf{r}})$ (that leave $\mathrm{Hom}(F)$ stable) of sequences of elements of $\mathrm{Hom}(F)$ which are incrementally bounded in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ with respect to some n_0 for which $f_{\mathbf{n}_0}$ belongs to $\mathbb{H}_{\Phi,\mathbf{r}}$.

Proposition 3. *Let $\overline{\mathrm{Hom}(F)}^{(-),\mathbf{b}^+}$ be as introduced in Definition 2.5, then,*

$$\overline{\mathrm{Hom}(F)}^{(-),\mathbf{b}^+} \bigcap \mathrm{Hom}(F) \subset \mathbb{H}_{\Phi,\mathbf{r}}. \tag{21}$$

Proof. Let $f \in \overline{\mathrm{Hom}(F)}^{(-),\mathbf{b}^+} \bigcap \mathrm{Hom}(F)$, we want to show that $|f|_{\Phi,\mathbf{r}}$ and $|f^{-1}|_{\Phi,\mathbf{r}}$ exist. By assumptions, there exists $\{f_{\mathbf{n}}\} \in \mathcal{IB}^+$, such that $\rho'_{\Phi,\mathbf{r}}(f, f_{\mathbf{n}}) \underset{\mathbf{n}\to\infty}{\longrightarrow} 0$. Consider n_0 such that $f_{\mathbf{n}_0} \in \mathbb{H}_{\Phi,\mathbf{r}}$ resulting from the definition of $\overline{\mathrm{Hom}(F)}^{(-),\mathbf{b}^+}$. From (14),

$$\begin{aligned}\rho'_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, Id_{\mathbf{F}}) \leq\ & a\beta\rho'_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, f_{\mathbf{n}_0})\rho'_{\Phi,\mathbf{r}}(f_{\mathbf{n}_0}, Id_{\mathbf{F}}) \\ & + (bm^{-1}\beta + \beta\gamma^{-1})\rho'_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, f_{\mathbf{n}_0}) + \rho'_{\Phi,\mathbf{r}}(f_{\mathbf{n}_0}, Id_{\mathbf{F}}).\end{aligned} \tag{22}$$

Since $\{f_{\mathbf{n}}\}$ is incrementally bounded in $\mathfrak{T}^+(\rho'_{\Phi,\mathbf{r}})$ with respect to n_0, then from (22), we deduce that the real-valued sequence $\{|f_{\mathbf{n}}|_{\Phi,\mathbf{r}}\}_{\mathbf{n}\geq\mathbf{n}_0}$ is bounded.

Besides, for any $x \in F$, and any $n \geq n_0$,

$$r(\|f(x) - x\|) \leq |ff_{\mathbf{n}}^{-1}|_{\Phi,\mathbf{r}}\Phi(f_{\mathbf{n}}(x)) + r(\|f_{\mathbf{n}}(x) - x\|),$$

and therefore by using the estimate (6) in Proposition 1,

$$\frac{r(\|f(x)-x\|)}{\Phi(x)} \leq |ff_{\mathbf{n}}^{-1}|_{\Phi,\mathbf{r}} \cdot \Lambda(|f_{\mathbf{n}}|_{\Phi,\mathbf{r}}) + |f_{\mathbf{n}}|_{\Phi,\mathbf{r}}, \tag{23}$$

since $\Lambda(|f_{\mathbf{n}}|_{\Phi,\mathbf{r}})$ is well defined for $n \geq n_0$ because $|f_{\mathbf{n}}|_{\Phi,\mathbf{r}}$ exist for such n. We get then trivially that $\{\Lambda(|f_{\mathbf{n}}|_{\Phi,\mathbf{r}})\}_{\mathbf{n}\geq n_0}$ is bounded because $\{|f_{\mathbf{n}}|_{\Phi,\mathbf{r}}\}_{\mathbf{n}\geq\mathbf{n}_0}$ is. Now since $|ff_{\mathbf{n}}^{-1}|_{\Phi,\mathbf{r}} \underset{\mathbf{n}\to\infty}{\longrightarrow} 0$ by assumption, we then deduce that $|f|_{\Phi,\mathbf{r}}$ exists by taking n sufficiently large in (23). To conclude, it suffices to note that (22) shows thanks to the incrementally bounded property, that $\{|f_{\mathbf{n}}^{-1}|_{\Phi,\mathbf{r}}\}_{\mathbf{n}\geq\mathbf{n}_0}$ is bounded as well, leading to the boundedness of $\{\Lambda(|f_{\mathbf{n}}^{-1}|_{\Phi,\mathbf{r}})\}_{\mathbf{n}\geq\mathbf{n}_0}$ which by repeating similar estimates leads to the existence of $|f^{-1}|_{\Phi,\mathbf{r}}$ by using the assumption: $|f^{-1}f_{\mathbf{n}}|_{\Phi,\mathbf{r}} \underset{\mathbf{n}\to\infty}{\longrightarrow} 0$. □

In the sequel we will need sometimes to use the following property verified by the function r. Let r satisfy the conditions of the preceding subsection and let G denote a continuous function $G : F \to E$, then for any K compact subset of F, there exists $x_{\mathbf{K}} \in K$, such that $r\Big(\sup_{\mathbf{x}\in\mathbf{K}} \|G(x)\|\Big) = r(\|G(x_{\mathbf{K}})\|) = \sup_{\mathbf{x}\in\mathbf{K}} r(\|G(x)\|)$, since r is increasing. Since we will need this property of r later, we make it precise as the condition,

$$(S) \quad : \text{ for all compact set } K \subset F, \; r\Big(\sup_{\mathbf{x}\in\mathbf{K}} \|G(x)\|\Big) = \sup_{\mathbf{x}\in\mathbf{K}} r(\|G(x)\|),$$

for every continuous function $G : F \to E$; condition which holds therefore for r as defined above. In what follows, $\rho_{\Phi,\mathbf{r}}$ will refer for both $\rho_{\Phi,\mathbf{r}}$ when applied to elements of $\mathbb{H}_{\Phi,\mathbf{r}}$, and to $\rho'_{\Phi,\mathbf{r}}$ when applied to elements of $\mathrm{Hom}(F)$, without any sort of confusion.

Let us now introduce the following concept of Cauchy sequence in $\mathbb{H}_{\Phi,\mathbf{r}}$ or more generally in $\mathrm{Hom}(F)$, which is adapted to our framework.

Definition 2.6. A sequence $\{f_{\mathbf{n}}\}$ in $\mathbb{H}_{\Phi,\mathbf{r}}$ or in $\mathrm{Hom}(F)$, is called $\rho^+_{\Phi,\mathbf{r}}$-Cauchy (resp. $\rho^-_{\Phi,\mathbf{r}}$-Cauchy), if the following condition holds,

$$\forall\, \epsilon > 0, \; \exists\, N \in \mathbb{N}, \; (p \geq q \geq N) \Rightarrow (\rho_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, f_{\mathbf{q}}) \leq \epsilon).$$

$$(\text{resp. } \forall\, \epsilon > 0, \; \exists\, N \in \mathbb{N}, \; (q \geq p \geq N) \Rightarrow (\rho_{\Phi,\mathbf{r}}(f_{\mathbf{p}}, f_{\mathbf{q}}) \leq \epsilon).$$

Remark 6. Note that, since $\rho_{\Phi,\mathbf{r}}$ is not symmetric, the role of p and q are not symmetric as well, to the contrary of the classical definition of a Cauchy sequence in a metric space.

Remark 7. By definition, every sequence $\{f_{\mathbf{n}}\}$ in $\mathbb{H}_{\Phi,\mathbf{r}}$ which is $\rho^+_{\Phi,\mathbf{r}}$−Cauchy (resp. $\rho^-_{\Phi,\mathbf{r}}$−Cauchy) belongs to $\mathcal{IB}^+_{\mathbf{u}}$ (resp. $\mathcal{IB}^-_{\mathbf{u}}$). However, a sequence $\{f_{\mathbf{n}}\}$ in $\mathrm{Hom}(F)$ which is $\rho^+_{\Phi,\mathbf{r}}$−Cauchy is not an element of $\mathcal{IB}^+_{\mathbf{u}}$ in general but is an element of $\mathcal{IB}^+$.

We are now in position to prove the following cornerstone proposition concerning the convergence in $\mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$ of $\rho^+_{\Phi,\mathbf{r}}$-Cauchy sequences in $\mathrm{Hom}(F)$. We recall that a topological space is defined to be σ-compact if and only if it is the union of a countable family of compact subsets [30].

Proposition 4. *Assume that F is an unbounded subset of E which is locally connected, σ-compact and locally compact*[5]*. Consider R given as in Definition 2.1, and $\Phi \in \mathcal{E}_F^R$. Let r satisfy the above assumptions including (A_r), and such that $(C_{r,R})$ is satisfied. Let $\{f_n\}$ be a sequence in $\mathrm{Hom}(F)$. If the following conditions hold:*

(C_1) *For every compact $K \subset F$, the sequence of the restriction of f_n to K, $\{f_n|_K\}$, is bounded. The same holds for the sequence of restrictions of the inverses, $\{f_n^{-1}|_K\}$,*

(C_2) *$\{f_n\}$ is $\rho^+_{\Phi,r}$-Cauchy,*

then $\{f_n\}$ converges in $\mathfrak{T}^-(\rho_{\Phi,r})$ towards an element of $\mathrm{Hom}(F)$.

If furthermore, either $\{f_n\}$ is a sequence of homeomorphisms living in $\mathbb{H}_{\Phi,r}$, or more generally $\{f_n\}$ is such that $f_{n_0} \in \mathbb{H}_{\Phi,r}$ for some n_0, then $\{f_n\}$ converges in $\mathfrak{T}^-(\rho_{\Phi,r})$ towards an element of $\mathbb{H}_{\Phi,r}$.

Proof. The proof is divided in several steps.

Step 1. Let $\{f_n\}$ be a sequence of homeomorphisms of F fulfilling the conditions of the proposition. Let $\epsilon > 0$ be fixed. Then from (C_2), there exists an integer N such that $p \geq q \geq N$ implies,

$$\begin{aligned}\forall\, x \in F,\ r(\|f_p(x) - f_q(x)\|) =& r(\|f_p f_q^{-1} f_q(x) - f_q(x)\|)\\ \leq& \rho_{\Phi,r}(f_p, f_q)\Phi(f_q(x)) \leq \epsilon\Phi(f_q(x)),\end{aligned} \tag{24}$$

which for every compact $K \subset F$, leads to,

$$\exists\, M_K > 0,\ \forall\, x \in K,\ r(\|f_p(x) - f_q(x)\|) \leq \epsilon M_K, \tag{25}$$

by assumption (C_1) and the continuity of Φ. Similarly, $p \geq q \geq N$, implies,

$$\begin{aligned}\forall\, x \in F,\ r(\|f_p^{-1}(x) - f_q^{-1}(x)\|) =& r(\|f_p^{-1} f_q f_q^{-1}(x) - f_q^{-1}(x)\|)\\ \leq& \rho_{\Phi,r}(f_p, f_q)\Phi(f_q^{-1}(x)) \leq \epsilon\Phi(f_q^{-1}(x)),\end{aligned} \tag{26}$$

which, for every compact $K \subset F$, can be summarized with (25), as,

$$\begin{aligned}&\exists\, M_K > 0,\ r\Big(\sup_{x\in K}\|f_p(x) - f_q(x)\|\Big) \leq \epsilon M_K,\ \text{and},\\ &r\Big(\sup_{x\in K}\|f_p^{-1}(x) - f_q^{-1}(x)\|\Big) \leq \epsilon M_K,\end{aligned} \tag{27}$$

by using (S) and labeling still by M_K the greater constant.

Recall from assumptions made on the function r in subsection 2.1, that the continuity of r holds at 0 and that $r(x) = 0$ if and only if $x = 0$. From that, we deduce that the sequences $\{f_n\}$ and $\{f_n^{-1}\}$ converge uniformly on each compact subset of F towards respectively a map $f : F \to F$ and a map $g : F \to F$. Note that by choosing appropriately a family of compact subsets of F, covering F, we get furthermore that f and g can be chosen continuous on F.

Step 2. Our main objective here is to show that $f = g^{-1}$, *i.e.* $f \in \mathrm{Hom}(F)$. Since F is assumed to be σ-compact and locally compact, there exists an exhaustive sequence of compacts sets $\{K_k\}_{k\in\mathbb{N}}$ of F; *e.g.* [2, Corollary 2.77]. From step 1, (27) is valid for any $p \geq q \geq N$, with N which does not depend on the compact K, and therefore we get that $\{f_n\}$ is Cauchy for the metric Δ on $\mathrm{Hom}(F)$, given by,

$$\Delta(\phi, \psi) = \delta(\phi, \psi) + \delta(\phi^{-1}, \psi^{-1}),$$

[5] Note that, since E is a finite dimensional normed vector space, which is locally compact Haussdorf space, if F is an open or closed subset of E, it is locally compact; cf. *e.g* [21, §3.18.4, p. 66].

where $\delta(\phi,\psi) := \sum_{\mathsf{k}=0}^{\infty} 2^{-\mathsf{k}} \|\phi-\psi\|_{\mathsf{k}}(1+\|\phi-\psi\|_{\mathsf{k}})^{-1}$, and $\|\phi-\psi\|_{\mathsf{k}} := \max_{\mathsf{x}\in \mathsf{K}_k} \|\phi(x)-\psi(x)\|$ for any compact K_{k}, and any homeomorphisms of F, ϕ and ψ.

Indeed, it suffices to note that for a given $\epsilon' > 0$, from (27) there exists l, ϵ and $N_{\square}$ such that,

$$\sum_{\mathsf{k}=\mathsf{l}+1}^{\infty} 2^{-\mathsf{k}} \leq \frac{\epsilon'}{4}, \text{ and } \sum_{\mathsf{k}=0}^{\mathsf{l}} 2^{-\mathsf{k}} \|f_{\mathsf{p}} - f_{\mathsf{q}}\|_{\mathsf{k}} \leq \frac{\epsilon'}{4},$$

which leads to $\delta(f_{\mathsf{p}}, f_{\mathsf{q}}) \leq \epsilon'/2$, and similarly to $\delta(f_{\mathsf{p}}^{-1}, f_{\mathsf{q}}^{-1}) \leq \epsilon'/2$, for any $p \geq q \geq N_{\square}$.

Now, since F is locally compact and locally connected from a famous result of Arens [3, Theorem 4], $\mathrm{Hom}(F)$ is complete for Δ which is a metric compatible with the compact-open topology [25], making $\mathrm{Hom}(F)$ a Polish group[6]. Therefore $\{f_{\mathsf{n}}\}$ converges in the compact-open topology towards an element $\mathfrak{h} \in \mathrm{Hom}(F)$. By recalling that the compact-open topology is here equivalent to the topology of compact convergence [44], we obtain by uniqueness of the limit that $f = \mathfrak{h} \in \mathrm{Hom}(F)$; f being the limit of $\{f_{\mathsf{n}}\}$ in the topology of compact convergence from step 1.

Step 3. Let us summarize what has been proved. We have shown under the assumptions (C_1) and (C_2), that we can produce from any sequence $\{f_{\mathsf{n}}\}$ which is $\rho^{+}_{\Phi,\mathsf{r}}$-Cauchy, an element $f \in \mathrm{Hom}(F)$, such that $\{f_{\mathsf{n}}\}$ converges uniformly to it on each compact subset of F, and $\{f_{\mathsf{n}}^{-1}\}$ does the same towards f^{-1}. In fact we can say more with respect to the topology of convergence.

Indeed, going back to (24), we have that,

$$\forall\, \epsilon > 0,\ \exists N \in \mathbb{N},\ (p \geq q \geq N) \Rightarrow \Big(\forall x \in F,\ r(\|f_{\mathsf{p}}(x) - f_{\mathsf{q}}(x)\|) \leq \epsilon \cdot \Phi(f_{\mathsf{q}}(x))\Big)$$

i.e. by making $p \to +\infty$ and using (A_{r}),

$$\forall\, \epsilon > 0,\ \exists N \in \mathbb{N},\ (q \geq N) \Rightarrow \Big(\forall x \in F,\ r(\|f(x) - f_{\mathsf{q}}(x)\|) \leq \epsilon \cdot \Phi(f_{\mathsf{q}}(x))\Big),$$

which leads to,

$$\forall\, \epsilon > 0,\ \exists N \in \mathbb{N},\ (q \geq N) \Rightarrow (|f f_{\mathsf{q}}^{-1}|_{\Phi,\mathsf{r}} \leq \epsilon).$$

From similar estimates, we get,

$$\forall\, \epsilon > 0,\ \exists N \in \mathbb{N},\ (q \geq N) \Rightarrow (|f^{-1} f_{\mathsf{q}}|_{\Phi,\mathsf{r}} \leq \epsilon).$$

We have thus shown the convergence of $\{f_{\mathsf{n}}\}$ in $\mathfrak{T}^{-}(\rho_{\Phi,\mathsf{r}})$, that is,

$$\rho_{\Phi,\mathsf{r}}(f, f_{\mathsf{q}}) \underset{\mathsf{q}\to+\infty}{\longrightarrow} 0.$$

At this stage, we have proved that $\{f_{\mathsf{n}}\}$ converges in $\mathfrak{T}^{-}(\rho_{\Phi,\mathsf{r}})$ towards an element of $\mathrm{Hom}(F)$.

Step 4. This last step is devoted to the proof of the last statement of the theorem concerning the membership of the limit of $\{f_{\mathsf{n}}\}$ to $\mathbb{H}_{\Phi,\mathsf{r}}$. This fact is simply a consequence of Proposition 2-(iv) in the case $\{f_{\mathsf{n}}\} \in (\mathbb{H}_{\Phi,\mathsf{r}})^{\mathbb{N}}$, and a consequence of Remark 7, and Proposition 3 in the case $\{f_{\mathsf{n}}\} \in (\mathrm{Hom}(F))^{\mathbb{N}}$, which gives in all

[6]Note that this reasoning is independent from the choice of the metric rendering $\mathrm{Hom}(F)$ complete, since it is known that $\mathrm{Hom}(F)$ has a unique Polish group structure (up to isomorphism); see [28].

the cases that the limit in $\mathfrak{T}^{-}(\rho_{\Phi,\mathbf{r}})$ of $\{f_{\mathbf{n}}\}$ lives in $\mathbb{H}_{\Phi,\mathbf{r}}$. The proof is therefore complete. □

Lastly, it is worth to note that it is only in step 3 of the preceding proof that was needed assumption ($A_{\mathbf{r}}$), but since Proposition 4 will be used in the proof of Theorems 3.1 and 4.3, we will make a systematic use of this assumption in the sequel.

3. A fixed point theorem in the homeomorphisms group.

In this section we state and prove a new fixed point theorem valid for self-mappings acting on $\mathrm{Hom}(F)$, which holds within the functional framework developed in the preceding section. This fixed point theorem uses a contraction mapping argument that involves a Picard scheme that has to be controlled appropriately due to the relaxed inequality (14). In this section $\rho_{\Phi,\mathbf{r}}$ will stand for the extended premetric introduced at the beginning of the subsection 2.2.

Theorem 3.1. *Consider R given as in Definition 2.1, and $\Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$, with F as in Proposition 4. Let r satisfy the above assumptions including ($A_{\mathbf{r}}$), and such that ($C_{\mathbf{r},\mathbf{R}}$) is satisfied. Let $\Upsilon : \mathrm{Hom}(F) \to \mathrm{Hom}(F)$ be an application. Let $\{f_{\mathbf{n}}\}$ be a sequence in $\mathrm{Hom}(F)$. We assume that there exists $h_0 \in \mathrm{Hom}(F)$ such that the following conditions hold:*

(i) *$\delta := \rho_{\Phi,\mathbf{r}}(\Upsilon(h_0), h_0) < A^{-1}$, where $A = \max(a\beta, bm^{-1}\beta + \beta\gamma^{-1})$.*
(ii) *$\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{Z}}$ is bounded on every compact of F.*

Assume furthermore that there exists a constant $0 < C < 1$, such that,

$$\forall\ (f,g)\ \in \mathrm{Hom}(F)\times\mathrm{Hom}(F),\ \rho_{\Phi,\mathbf{r}}(\Upsilon(f),\Upsilon(g)) < C\rho_{\Phi,\mathbf{r}}(f,g), \tag{28}$$

then there exists a unique $h \in \mathrm{Hom}(F)$ such that $\Upsilon(h) = h$, which is obtained as a limit in $\mathfrak{T}^{-}(\rho_{\Phi,\mathbf{r}})$ of $\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$.

Furthermore, if there exists $n_0 \in \mathbb{N}$ such that $\Upsilon^{\mathbf{n}_0}(h_0) \in \mathbb{H}_{\Phi,\mathbf{r}}$, then $h \in \mathbb{H}_{\Phi,\mathbf{r}}$.

Remark 8. Note that $A > 1$ since $\beta > \gamma$ by assumption (G_3).

Proof. From Proposition 4, it suffices to show that $\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$ is $\rho^{+}_{\Phi,\mathbf{r}}$-Cauchy for any $h_0 \in \mathrm{Hom}(F)$ satisfying (i). For simplifying the notations we denote by $\rho_{\mathbf{n}}^{\mathbf{m}}$ the quantity $\rho_{\Phi,\mathbf{r}}(\Upsilon^{\mathbf{m}}(h_0), \Upsilon^{\mathbf{n}}(h_0))$, for $m \geq n$. Note that since δ is finite, by recurrence and using the contraction property (28) we can show that all the quantities $\rho_{\mathbf{n}}^{\mathbf{m}}$ are finite as well.

By using the relaxed inequality (14), the contraction property (28) and the definition of A in condition (i), it is easy to obtain for any integers $n, m \geq n+1$, and $k \geq 1$,

$$\begin{aligned} \rho_{\Phi,\mathbf{r}}(\Upsilon^{\mathbf{m}+\mathbf{k}}(h_0), \Upsilon^{\mathbf{n}}(h_0)) \leq A\delta C^{\mathbf{m}+\mathbf{k}-1}\rho_{\Phi,\mathbf{r}}(\Upsilon^{\mathbf{m}+\mathbf{k}-1}(h_0), \Upsilon^{\mathbf{n}}(h_0)) \\ + A\delta C^{\mathbf{m}+\mathbf{k}-1} + \rho_{\Phi,\mathbf{r}}(\Upsilon^{\mathbf{m}+\mathbf{k}-1}(h_0), \Upsilon^{\mathbf{n}}(h_0)), \end{aligned} \tag{29}$$

which leads to,

$$\rho_{\mathbf{n}}^{\mathbf{m}+\mathbf{k}} < C^{\mathbf{m}+\mathbf{k}-1}\rho_{\mathbf{n}}^{\mathbf{m}+\mathbf{k}-1} + C^{\mathbf{m}+\mathbf{k}-1} + \rho_{\mathbf{n}}^{\mathbf{m}+\mathbf{k}-1}, \tag{30}$$

by using $A\delta < 1$ from assumption (i) and the notations specified above.

Let $\epsilon > 0$ be fixed. For any m and n, we introduce now the two-parameters sequence $\{F_{\mathbf{k}}(m,n)\}_{\mathbf{k}\in\mathbb{N}}$ defined by recurrence through,

$$\begin{cases} F_{\mathbf{k}}(m,n) = C^{\mathbf{m}+\mathbf{k}-1}F_{\mathbf{k}-1}(m,n) + C^{\mathbf{m}+\mathbf{k}-1} + F_{\mathbf{k}-1}(m,n), \forall\, k \geq 1, \\ F_0(m,n) = \epsilon. \end{cases} \tag{31}$$

When no ambiguity is possible, $F_{\mathbf{k}}$ will simply stand for $F_{\mathbf{k}}(m,n)$. The role of m and n will be apparent in a moment.

Moreover, from (30), it is easy to show that for any n and $m \geq n+1$,

$$(\rho_{\mathbf{n}}^{\mathbf{m}} \leq F_0(m,n)) \Rightarrow (\forall\, k \geq 1,\ \rho_{\mathbf{n}}^{\mathbf{m}+\mathbf{k}} \leq F_{\mathbf{k}}(m,n)).$$

As we will see, it suffices to consider $m = n+1$ to prove the theorem, a choice that we make in what follows. Since $C < 1$, obviously,

$$\exists\, N_1 \ :\ \forall\, n \geq N_1, \rho_{\mathbf{n}}^{\mathbf{n}+1} < C^{\mathbf{n}}\delta < \epsilon = F_0(n+1,n),$$

and therefore we get that $\rho_{\mathbf{n}}^{\mathbf{n}+1+\mathbf{k}} \leq F_{\mathbf{k}}(n+1,n)$, from what precedes.

The key idea is now to note that if for all $k \geq 0$, $F_{\mathbf{k}}(n+1,n) \leq 2\epsilon$, for n sufficiently big, then the sequence $\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$ is $\rho^{+}_{\Phi,\mathbf{r}}$-Cauchy for $h_0 \in \mathrm{Hom}(F)$ fulfilling condition (i). In the sequel we prove that it is indeed the case.

To do so, an easy recurrence shows that,

$$\forall\, k \in \mathbb{N},\ F_{\mathbf{k}} > 0, \text{and } \{F_{\mathbf{k}}\} \text{ is strictly increasing.}$$

In particular,

$$\forall\, k \in \mathbb{N},\ F_{\mathbf{k}} \geq \epsilon,$$

and therefore for any $k \geq 1$ and m,

$$\frac{F_{\mathbf{k}}}{F_{\mathbf{k}-1}} = C^{\mathbf{m}+\mathbf{k}-1} + 1 + \frac{C^{\mathbf{m}+\mathbf{k}-1}}{F_{\mathbf{k}-1}} \leq C^{\mathbf{m}+\mathbf{k}-1}\big(1+\frac{1}{\epsilon}\big) + 1. \tag{32}$$

Thus by using (31) and iterating (32), we obtain for any $k \geq 1$,

$$\begin{aligned} v_{\mathbf{k}} := F_{\mathbf{k}} - F_{\mathbf{k}-1} &< C^{\mathbf{m}+\mathbf{k}-1}(F_{\mathbf{k}-1}+1) \\ &\leq C^{\mathbf{m}+\mathbf{k}-1}\Big\{\prod_{\mathbf{l}=2}^{\mathbf{k}}\Big(C^{\mathbf{m}+\mathbf{l}-2}(1+\frac{1}{\epsilon})+1\Big)\cdot\epsilon+1\Big\}, \end{aligned} \tag{33}$$

with the convention $\prod_{\mathbf{l}=2}^{\mathbf{k}=1}\Big(C^{\mathbf{m}+\mathbf{l}-2}(1+\frac{1}{\epsilon})+1\Big) \equiv 1$, making valid (33) for $k=1$.

Since $C < 1$, then for any $l \in \{2,...,k\}$, and $k \geq 2$, $C^{\mathbf{m}+\mathbf{l}-2} \leq C^{\mathbf{m}}$, which leads from (33) to,

$$v_{\mathbf{k}} \leq C^{\mathbf{m}}\Big\{\Big(C^{\mathbf{m}+1}\big(1+\epsilon^{-1}\big)+C\Big)^{\mathbf{k}-1}\cdot\epsilon + C^{\mathbf{k}-1}\Big\}, \tag{34}$$

which is also valid for $k=1$, by simply computing $F_1 - F_0$.

Besides,

$$\exists\, N_2 \ :\ \forall\, m \geq N_2,\ a_{\mathbf{m}} := C^{\mathbf{m}+1}(1+\epsilon^{-1}) + C < 1, \tag{35}$$

which shows for $m \geq N_2$,

$$\sum_{\mathbf{j}=1}^{\mathbf{j}=\mathbf{k}} v_{\mathbf{j}} \leq C^{\mathbf{m}}\Big\{\frac{\epsilon}{1-a_{\mathbf{m}}} + \frac{1}{1-C}\Big\}, \tag{36}$$

since $C < 1$ and $a_{\mathbf{m}} < 1$. Now it can be shown,

$$\exists\, N_3 \ :\ \forall\, m \geq N_3,\ C^{\mathbf{m}}\Big\{\frac{\epsilon}{1-a_{\mathbf{m}}} + \frac{1}{1-C}\Big\} \leq \epsilon. \tag{37}$$

Fixing $m = n+1$, we conclude from (36) and the trivial identity $F_{\mathbf{k}} = \sum_{\mathbf{j}=1}^{\mathbf{j}=\mathbf{k}} v_{\mathbf{j}} + F_0$ that,

$$\forall\, n \in \mathbb{N}, \forall k \in \mathbb{N},\ (n \geq \max(N_1, N_2, N_3)) \Rightarrow (F_{\mathbf{k}}(n+1, n)) \leq 2\epsilon), \tag{38}$$

which shows in particular that,

$$\forall\, n \in \mathbb{N}, \forall k \in \mathbb{N},\ (n \geq \max(N_1, N_2, N_3)) \Rightarrow (\rho_{\mathbf{n}}^{\mathbf{n}+\mathbf{k}+1} \leq 2\epsilon). \tag{39}$$

We have thus proved that $\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$ is $\rho_{\Phi,\mathbf{r}}^{+}$-Cauchy for any $h_0 \in \mathrm{Hom}(F)$ fulfilling conditions (i) and (ii), and thus by Proposition 4, $h := \lim_{\mathbf{n}\to\infty} \Upsilon^{\mathbf{n}}(h_0)$ exists in $\mathfrak{T}^{-}(\rho_{\Phi,\mathbf{r}})$.

It can be shown furthermore that, for every $h_0 \in \mathrm{Hom}(F)$, $\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$ is incrementally bounded with respect to any $n_0 \in \mathbb{N}$, due to the contraction property and the fact that $\{\Upsilon^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$ is $\rho_{\Phi,\mathbf{r}}^{+}$-Cauchy. Consequently, if there exists n_0 such that $\Upsilon^{\mathbf{n}_0}(h_0) \in \mathbb{H}_{\Phi,\mathbf{r}}$ then by applying Proposition 4 again (last part) we obtain $h \in \mathbb{H}_{\Phi,\mathbf{r}}$. □

4. **A conjugacy theorem and the generalized spectrum of the Koopman operator.**

4.1. **The conjugacy theorem.**

We prove in this section the main result of this article, *i.e* the conjugacy Theorem 4.3. To do so, we need further preliminary tools and notations that we describe hereafter. In this section we assume the previous assumptions on r (see subsection 2.1) including the condition (A$_{\mathbf{r}}$). As in Section 3 the premetric $\rho_{\Phi,\mathbf{r}}$ will stand for the extended premetric introduced in subsection 2.2. In what follows, we endow again $\mathrm{Hom}(F)$ with a topology $\mathfrak{T}^{-}(\rho_{\Phi,\mathbf{r}})$ where r and R satisfy the condition (C$_{\mathbf{r},\mathbf{R}}$) of subsection 2.1, except that here Φ belonging to $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ is not arbitrary and has to solve a generalized eigenvalue problem to handle the conjugacy problem; cf Theorem 4.3.

For any self-mapping f of F, we introduce the following r-Lipschitz constant,

$$\lambda_{\mathbf{r}}(f) := \sup\Big\{\frac{r(\|f(x) - f(y)\|)}{r(\|x-y\|)},\ x, y \in F,\ x \neq y\Big\}, \tag{40}$$

which can be infinite. This quantity is clearly a direct extension of the classical notion of Lipschitz constant, $\mathrm{Lip}(f)$, of a function f on normed vector space where the norm, $\|\cdot\|$, has been substituted by the subadditive map $r(\|\cdot\|)$.

Definition 4.1. We denote by $\mathbb{L}_{\mathbf{r}}(F)$ the set of homeomorphisms of F such that $\lambda_{\mathbf{r}}(f)$, and $\lambda_{\mathbf{r}}(f^{-1})$ exist. Such an homeomorphism is called an r-Lipeomorphism of F.

We will also need the following concept of generalized eigenvalue of the Koopman operator, outlined in the introduction.

Definition 4.2. Let $f \in \mathrm{Hom}(F)$, and $U_{\mathbf{f}}$ its Koopman operator with domain $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$. A generalized eigenvalue of $U_{\mathbf{f}}$ is any $\lambda \in \mathbb{R}$ such that,

$$\exists\, \Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}\ :\ U_{\mathbf{f}}(\Phi) \geq \lambda\Phi, \tag{41}$$

where in case of existence, Φ is the corresponding generalized eigenfunction.

Remark 9. Note that if $f \in \mathbb{H}_{\Phi,\mathbf{r}}$ then $U_{\mathbf{f}}(\Phi) \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ for any generalized eigenfunction Φ, from Proposition 1 and Remark 3. The case of equality makes thus sense in (41), justifying *de facto* the terminology. When $f \in \mathrm{Hom}(F)\backslash\mathbb{H}_{\Phi,\mathbf{r}}$, note that the generalized eigenvalue problem (41) may still exhibit solutions in some appropriate space $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ such that $U_{\mathbf{f}}(\Phi) \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$, as illustrated in subsection 4.2.

Lastly, given f and g in $\mathrm{Hom}(F)$, we introduce the following classical *conjugacy operator*,

$$\mathcal{L}_{\mathbf{f},\mathbf{g}} : \begin{cases} \mathrm{Hom}(F) \to \mathrm{Hom}(F) \\ h \to \mathcal{L}_{\mathbf{f},\mathbf{g}}(h) := f \circ h \circ g^{-1}. \end{cases} \tag{42}$$

We are now in a position to state and prove the main theorem of this article, a conjugacy theorem which is conditioned to a generalized eigenvalue problem of the related Koopman operators.

Theorem 4.3. *Given f and g in $\mathrm{Hom}(F)$ where F is as in Proposition 4, assume that there exist a growth function R given as in Definition 2.1 and a function r satisfying the above assumptions including ($A_{\mathbf{r}}$), such that ($C_{\mathbf{r},\mathbf{R}}$) is satisfied; and such that the following conditions are fulfilled,*

(a) $f, g \in \mathbb{L}_{\mathbf{r}}(F)$,
(b) *there exists $\alpha > 1$ and a common generalized eigenfunction $\Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ of the Koopman operators $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$, which solves the following generalized eigenvalue problem,*

$$\mathbf{P}_{\alpha} : \begin{cases} U_{\mathbf{f}}(\Phi) \geq \alpha\lambda_{\mathbf{r}}(f)\Phi, \\ U_{\mathbf{g}}(\Phi) \geq \alpha\lambda_{\mathbf{r}}(g)\Phi. \end{cases} \tag{43}$$

Under these conditions, for any Φ solving (43), assume further that there exists an homeomorphism h_0 of F satisfying the following properties:

(i) $\delta := \rho_{\Phi,\mathbf{r}}(\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_0), h_0) < A^{-1}$, *where* $A = \max(a\beta, bm^{-1}\beta + \beta\gamma^{-1})$.
(ii) $\{\mathcal{L}_{\mathbf{f},\mathbf{g}}^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{Z}}$ *is bounded on every compact of F.*

Then f and g are conjugated by a unique element h of $\mathrm{Hom}(F)$, which is the limit in $\mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$ of $\{\mathcal{L}_{\mathbf{f},\mathbf{g}}^{\mathbf{n}}(h_0)\}_{\mathbf{n}\in\mathbb{N}}$. Furthermore, if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{L}_{\mathbf{f},\mathbf{g}}^{\mathbf{n}_0}(h_0) \in \mathbb{H}_{\Phi,\mathbf{r}}$, then $h \in \mathbb{H}_{\Phi,\mathbf{r}}$.

Proof. Let f and g be two homeomorphisms of F. Let the function r and the growth function R be such that the conditions (a) and (b) of Theorem 4.3 are satisfied. Let $\Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ be a solution of (43). We endow then $\mathrm{Hom}(F)$ with the topology $\mathfrak{T}^-(\rho_{\Phi,\mathbf{r}})$ for such a Φ.

Since the existence of a solution in $\mathrm{Hom}(F)$ to the conjugacy problem is equivalent to the existence of a fixed point in $\mathrm{Hom}(F)$ of $\mathcal{L}_{\mathbf{f},\mathbf{g}}$, it suffices from Theorem 3.1 to examine if $\mathcal{L}_{\mathbf{f},\mathbf{g}}$ is a contraction in the premetric $\rho_{\Phi,\mathbf{r}}$. To do so, we need to estimate $|\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_1) \circ (\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_2))^{-1}|_{\Phi,\mathbf{r}}$ as well as $|(\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_1))^{-1} \circ \mathcal{L}_{\mathbf{f},\mathbf{g}}(h_2)|_{\Phi,\mathbf{r}}$ for all $h_1, h_2 \in \mathrm{Hom}(F)$. Simple computations show that for all $x \in F$,

$$\frac{r(\|f \circ h_1 \circ h_2^{-1} \circ f^{-1}(x) - x\|)}{\Phi(x)} \leq \lambda_{\mathbf{r}}(f)\frac{r(\|h_1 \circ h_2^{-1} \circ f^{-1}(x) - f^{-1}(x)\|)}{\Phi(x)}, \tag{44}$$

which leads to,

$$|\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_1) \circ (\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_2))^{-1}|_{\Phi,\mathbf{r}} \leq \lambda_{\mathbf{r}}(f) \cdot \sup_{\mathbf{u}\in\mathbf{F}}\Big(\frac{\Phi(u)}{\Phi(f(u))}\Big) \cdot |h_1 \circ h_2^{-1}|_{\Phi,\mathbf{r}} \ . \tag{45}$$

Similar computations show,

$$|(\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_1))^{-1} \circ \mathcal{L}_{\mathbf{f},\mathbf{g}}(h_2)|_{\Phi,\mathbf{r}} \leq \lambda_{\mathbf{r}}(g) \cdot \sup_{\mathbf{u}\in\mathbf{F}}\Big(\frac{\Phi(u)}{\Phi(g(u))}\Big) \cdot |h_1^{-1} \circ h_2|_{\Phi,\mathbf{r}} \ , \tag{46}$$

and since Φ solves the generalized eigenvalue problem $\mathbf{P}_{\boldsymbol{\alpha}}$, we get for all $u \in F$,

$$\lambda_{\mathbf{r}}(f)\frac{\Phi(u)}{\Phi(f(u))} \leq \frac{1}{\alpha} < 1, \text{ and } \lambda_{\mathbf{r}}(g)\frac{\Phi(u)}{\Phi(g(u))} \leq \frac{1}{\alpha} < 1, \tag{47}$$

which allows us to conclude that,

$$\rho_{\Phi,\mathbf{r}}(\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_1), \mathcal{L}_{\mathbf{f},\mathbf{g}}(h_2)) \leq \frac{1}{\alpha} \cdot \rho_{\Phi,\mathbf{r}}(h_1, h_2), \tag{48}$$

for all h_1 and h_2 in $\mathrm{Hom}(F)$, *i.e.* the conjugacy operator $\mathcal{L}_{\mathbf{f},\mathbf{g}}$ is a contraction for the premetric $\rho_{\Phi,\mathbf{r}}$. The rest of the assumptions (i) and (ii) of Theorem 4.3 are just a translation of the ones used in Theorem 3.1, and thus by using this last theorem the proof of the present one is easily achieved. □

Remark 10. This theorem provides conditions for the conjugacy h, when it exists, to lie in $\mathbb{H}_{\Phi,\mathbf{r}}$. This means in such a case that h satisfies some behavior at infinity prescribed by Φ, which has in turn to solve $\mathbf{P}_{\boldsymbol{\alpha}}$, a spectral problem related to the Koopman operators $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$, which involves structural constants of f and g: $\lambda_{\mathbf{r}}(f)$ and $\lambda_{\mathbf{r}}(g)$ as introduced above. This aspect could be of interest in control theory.

Remark 11. Condition of type (i) in Theorem 4.3 is often met in a stronger form when dealing with (local) conjugacy problems that arise around a fixed point. Indeed it is often required that the convergence of the sequence $\{f^{\mathbf{n}}g^{-\mathbf{n}}\}$ holds in C^0 provided that f and g are tangent to sufficiently high order; cf. *e.g.* [13, Lemma 3 p. 95].

4.2. **An illustrative example.** To simplify we set $E = \mathbb{R}$ and we consider $F = [0, +\infty)$ which fulfills the conditions of Theorem 4.3. We consider furthermore $r(x) = x$ and $\Phi(x) = R(x) = \sqrt{x} + 1$ (for $x \in F$) that are subadditive function on F. Note that such an R satisfies the conditions of subsection 2.1, and that $\Phi \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ with $m = 1$ in (G_1) and, for instance, $\beta = 2$ and $\gamma = 1/2$ for (G_3). Note also that r as continuous function satisfies ($\mathrm{A}_{\mathbf{r}}$) (and (S)) and fulfill all the other standing assumptions. The cross condition ($\mathrm{C}_{\mathbf{r},\mathbf{R}}$) of subsection 2.1 is trivially fulfilled, since for instance:

$$1 + \sqrt{u} \leq u + \frac{5}{4}, \ \forall\, u \in \mathbb{R}^+.$$

We consider the following dynamics on F: $f(x) = \eta x$ with $0 < \eta < 1$ and $g(x) = \eta x + \varphi(x)$, where $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with compact support is such that g is an homeomorphism of F; φ will be further characterized in a moment. Other conditions on φ will be imposed below. Lastly note that $\lambda_{\mathbf{r}}(f) = \mathrm{Lip}(f)$ for $r(x) = x$. Since $\eta < 1$, there exists $\epsilon > 0$ such that,

$$\frac{\sqrt{\eta}}{\eta} > 1 + \epsilon. \tag{49}$$

Now since $\eta(1 + \epsilon) \leq \sqrt{\eta} < 1$, we get for all $x \in F$,

$$\alpha \mathrm{Lip}(f)\Phi(x) = \eta(1 + \epsilon)(\sqrt{x} + 1) \leq \sqrt{\eta x} + 1 = \Phi(f(x)), \tag{50}$$

with $\alpha = 1 + \epsilon$, which shows that Φ is a generalized eigenfunction in $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ of $U_{\mathbf{f}}$ with eigenvalue $\lambda = (1 + \epsilon)\mathrm{Lip}(f)$ in this particular context.

From (49) we get,

$$\exists\, \epsilon_2 > 0, : \ \frac{\sqrt{\eta}}{\eta + \epsilon_2} > (1 + \epsilon), \text{ and } \epsilon_2 < \eta. \tag{51}$$

Besides, for such an ϵ_2, there exists a function $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with compact support such that,

$$\mathrm{Lip}(g) = \max_{\mathbf{x}\in\mathbf{F}} |\eta + \varphi'(x)| = \eta + \epsilon_2, \tag{52}$$

and such that g is still an homeomorphism of F (since $\epsilon_2 < \eta$).

From (52) and (51) we get now,

$$\begin{aligned}(1+\epsilon)\mathrm{Lip}(g)\Phi(x) &= (1+\epsilon)(\eta+\epsilon_2)(\sqrt{x}+1) \\ &\leq \sqrt{\eta x} + 1 \leq \sqrt{\eta x + \varphi(x)} + 1 = \Phi(g(x)),\end{aligned} \tag{53}$$

which shows that Φ is a generalized eigenfunction in $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ of $U_{\mathbf{g}}$ with $\lambda = (1+\epsilon)\mathrm{Lip}(g)$. We are then left with a common eigenfunction of $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$ satisfying $\mathbf{P}_{\boldsymbol{\alpha}}$ with $\alpha = 1+\epsilon$. From our assumptions, it is easy to check furthermore that f and g belong to $\mathbb{L}_{\mathbf{r}}(F)$, and therefore conditions (a) and (b) of Theorem 4.3 are satisfied in this particular setting. Recall from the proof of Theorem 4.3 that these conditions ensure the contraction of the conjugacy operator $\mathcal{L}_{\mathbf{f},\mathbf{g}}$ for the premetric $\rho_{\Phi,\mathbf{r}}$.

To apply Theorem 4.3 we have now to check the remaining conditions (i) and (ii) for a convenient $h_0 \in \mathrm{Hom}(F)$. Let us take $h_0 = g$. We check first condition (i). In that respect, we have to estimate $|\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_0)\circ h_0^{-1}|_{\Phi,\mathbf{r}} = |fg^{-1}|_{\Phi,\mathbf{r}}$ and $|(\mathcal{L}_{\mathbf{f},\mathbf{g}}(h_0))^{-1}\circ h_0|_{\Phi,\mathbf{r}} = |f^{-1}g|_{\Phi,\mathbf{r}}$ since $h_0 = g$. Note that there exists $\psi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $g^{-1}(x) = x/\eta + \psi(x)$, and ψ has compact support. Note also that we can find φ and thus ψ such that,

$$\nu := \max\Big(\max_{\mathbf{x}\in\mathbf{F}}|\varphi|, \max_{\mathbf{x}\in\mathbf{F}}|\psi|\Big) < \eta A^{-1},$$

without violating (52) and thus having Φ still satisfying $\mathbf{P}_{\boldsymbol{\alpha}}$. For such a choice, $\eta\nu < \eta^2 A^{-1} < A^{-1}$ since $\eta < 1$, and in particular,

$$|fg^{-1}|_{\Phi,\mathbf{r}} = \sup_{\mathbf{x}\in\mathbf{F}}\Big(\frac{|\eta\psi(x)|}{\sqrt{x}+1}\Big) < A^{-1}, \tag{54}$$

and,

$$|f^{-1}g|_{\Phi,\mathbf{r}} = \sup_{\mathbf{x}\in\mathbf{F}}\Big(\frac{|\varphi(x)|}{\eta(\sqrt{x}+1)}\Big) < A^{-1}, \tag{55}$$

which allows us to conclude that condition (i) is checked with $h_0 = g$.

Finally let us check condition (ii). Since φ and ψ have compact supports then it can be shown that for $h_0 = g$, the sequence $\{\mathcal{L}^{\mathbf{n}}_{\mathbf{f},\mathbf{g}}(h_0)\}_{\mathbf{n}\in\mathbb{Z}}$ is bounded on every compact subset of F. We leave the details to the reader. We can thus apply Theorem 4.3 to conclude that f and g are conjugated which was of course obvious for $\mathrm{Lip}(\varphi)$ sufficiently small, from a trivial application of the global Hartman-Grobman theorem, cf. for instance [26].

This modest example is just intended to illustrate some mechanisms of the approach developed in this article. Of course, further investigations are needed with respect to the existence of solutions of the generalized eigenvalue problem $\mathbf{P}_{\boldsymbol{\alpha}}$ in spaces of type $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$, for more general homeomorphisms. We postpone this difficult task for a future work, discussing in the next subsection some related issues.

4.3. **Generalized eigenfunctions of the Koopman operator.** We describe here two possible approaches to examine the generalized eigenvalue problem $\mathbf{P}_{\boldsymbol{\alpha}}$, the first one is based on Schröder equations and the second one is based on cohomological equations. The point of view retained is based on functional equations techniques coming from different part of that literature where we emphasize the overlapping. In both cases, by shortly reviewing the existing results, we provide hereafter conditions under which the generalized eigenvalue problem $\mathbf{P}_{\boldsymbol{\alpha}}$ may possess continuous solutions, without being able to specify — in a general setting — if Φ can live in some space $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$. We elaborate on this point here with the intent of gathering some results related to our problem which are found dispersed in the literature. Note that in the sequel, we will focus more precisely on the generalized eigenvalue problem for $U_{\mathbf{f}}$ and not $\mathbf{P}_{\boldsymbol{\alpha}}$ itself, in order to exhibit already the main issues for the associated single existence problem of a generalized eigenfunction.

4.3.1. *Approach based on Schröder equations.* We recall first some background concerning Schröder equation. Here E denotes a real or complex normed vector space, $\mathcal{H}$ denotes an arbitrary space of $\mathbb{R}-$ or $\mathbb{C}-$valued functions on E, and $\mathcal{F}$ denotes some space of self mappings of E. Let f be in $\mathcal{F}$, then the Schröder's equation in $\mathcal{H}$ is the equation of unknown Ψ (to be found in $\mathcal{H}$):

$$\Psi \circ f = \lambda \cdot \Psi, \tag{56}$$

for some λ. It is the equation related to the spectrum of the Koopman operator[7] $\mathrm{U}_{\mathbf{f}} : \Psi \mapsto \Psi \circ f$, in the space $\mathcal{H}$. The properties of this spectrum are closely related to the function f as well the space $\mathcal{H}$. The Schröder's equation has a long history and has been extended and studied in various settings. In the early 1870s, Ernst Schröder [47] studied this type of functional equation in the complex plane for the composition operator, for $\Psi(z) = z^2$ and $f(z) = z + 1$. The functional equation named after him is $\Psi \circ f = \lambda\Psi$ where f is a given complex function, and the problem consists of finding Ψ and λ to satisfy the equation, *i.e.* an eigenvalue problem for $U_{\mathbf{f}}$. An important part of the results in the literature are devoted to contexts where f is a function mapping the unit disc in the complex plane onto itself initiated by the seminal work of Gabriel Koenigs in 1884 [31]. The reader may consult [15] or [50] with references therein, for a recent account about this part of the literature. This functional problem has also been considered historically for maps of the half-line [34] or more general Banach spaces in the past decades [53]. We mention lastly, that the Schröder equation is sometimes encountered under the form of the Poincaré functional equation [27, 34] and arises in various applications such as iterated function theory [20, 35], branching process [48] or dynamical systems theory [10, 54].

Naturally, the generalized eigenvalue problem $\mathbf{P}_{\boldsymbol{\alpha}}$ can be related to Schröder equations. Indeed, if the following Schröder equation,

$$\Psi \circ f = \alpha\lambda_{\mathbf{r}}(f) \cdot \Psi, \tag{57}$$

has a solution $\Psi : F \to F$ for $\alpha > 1$, then the generalized eigenvalue problem for $U_{\mathbf{f}}$ has an obvious solution, provided that $\Phi(\cdot) = \|\Psi(\cdot)\| \in \mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ and $f \in \mathbb{L}_{\mathbf{r}}(F)$. It is known that such an equation can be solved for particular domain F and particular space of functions over F such as Hardy spaces; see [18]. It is interesting to note that most of the results typically require some compactness assumptions of the Koopman operator $U_{\mathbf{f}}$ which involve that f possesses at least a fixed point in F; cf. [12, 49], cf. also [15, Theorem 5.1] for extensions of results of [12].

There exist other techniques coming from functional analysis rather than complex analysis, to deal with $\mathbf{P}_{\boldsymbol{\alpha}}$ from the point of view of Schröder equations. It consists of considering a more general type of Schröder equations where the unknown is a

[7]The Koopman operator is also known as the composition operator in other fields [18].

map $\Psi : F \to E$ aiming to satisfy,

$$U_{\mathbf{f}}(\Psi) = A \circ \Psi, \tag{58}$$

where A is a linear map of E. If we assume that A is invertible, and that there exist a C^0-functional $\mathcal{N} : E \to \mathbb{R}$ with a constant $\mathfrak{m} > 0$ satisfying,

$$\forall\, x \in F, \quad \mathcal{N}(f(x)) \geq \mathcal{N}(x) + \mathfrak{m}, \tag{59}$$

then [53, Theorem 2.1] permits to conclude the existence of a *continuous nonzero* solution of (58). Note that if $F \subset E \backslash \{0\}$ and $\|f(x)\| \leq \kappa \|x\|$ on F, with $\kappa \in (0, 1)$ then the functional inequality (59) is satisfied on F by simply taking $\mathcal{N}(x) := -\log \|x\|$ and $\mathfrak{m} := -\log(\kappa)$.

Now by assuming A invertible,

$$\forall\, \xi \in E, \ \|\xi\| \leq \|A^{-1}\| \|A\xi\|,$$

$$i.e., \ \|A^{-1}\|^{-1} \leq \inf_{\boldsymbol{\xi} \in \mathbf{E}-\{0\}} \frac{\|A\xi\|}{\|\xi\|},$$

and thus if,

$$\|A^{-1}\|^{-1} \geq \alpha \lambda_{\mathbf{r}}(f), \tag{60}$$

and there exists a couple $(\mathfrak{m}, \mathcal{N})$ satisfying (59), we obtain that $\Phi(\cdot) := \|\Psi(\cdot)\|$ with Ψ a solution of (58), is a solution of $U_{\mathbf{f}}(\Phi) \geq \alpha \lambda_{\mathbf{r}}(f) \cdot \Phi$. Thus in order to have a solution of the generalized eigenvalue problem for $U_{\mathbf{f}}$, we only need to know about the asymptotic behavior of Φ. However, the examination of the growth of an eigenfunction Ψ solution of (58) is a difficult task in general, [8] which renders the generalized eigenvalue problem for $U_{\mathbf{f}}$ and thus the functional problem $\mathbf{P}_{\boldsymbol{\alpha}}$ introduced here non-trivial to solve in general.

To conclude, we emphasize that the Schröder equation is related to Abel's functional equation [1], which is a well known functional equation often presented into the form $\varphi(f(x)) = \varphi(x) + 1$, where $\varphi : X \to \mathbb{C}$ is an unknown function and $f : X \to X$ is a given continuous mapping of a topological space X [34].[9] If the Abel equation possesses a continuous solution, then it provides a continuous solution to (59). Thus Abel's equation has a central role in our approach. As the next subsection indicates, there are deep connections between both equations and our functional problem.

4.3.2. *Approach based on cohomology equations.* Even if, to the best of the knowledge of the authors, Theorem 4.3 exhibits new relations between the existence of a conjugacy between two homeomorphisms and the spectrum of the related Koopman operators, relations between conjugacy problems and functional equations are far to be new. They arise classically under the form of the *Livshitz cohomology equation* [37, 38], $\phi = \Phi \circ f - \Phi$, where $f : \mathfrak{M} \to \mathfrak{M}$ is a dynamical system of some manifold $\mathfrak{M}$; $\phi : \mathfrak{M} \to \mathbb{R}$, a given function, and Φ maps $\mathfrak{M}$ into $\mathbb{R}$ or a multidimensional space; see for instance [6, 7, 19, 29]. As pointed by Livshitz [37, 38] the existence of a continuous solution Φ strictly depends on the dynamics generated by f and the topological as well as geometrical properties of $\mathfrak{M}$. For instance if we consider the particular case of the Abel equation, $\Phi(f(x)) = \Phi(x) + 1$, there is no continuous

[8] *e.g.* [11, 49] or [4] for particular cases related to the standard Schröder equation (57).

[9] The link between the both is trivial when $X \equiv \mathbb{C}$ where every solution φ of the Abel equation leads to a solution $\Psi : x \mapsto \exp(\log(\lambda)\varphi(x))$ for every $\lambda > 0$ of the Schröder equation. In such a case the spectrum of U_f contains $(0, +\infty)$.

solution if $\mathfrak{M}$ is compact, since if such solution would exist, $\Phi(f^{\mathbf{n}}(x)) = \Phi(x) + n$, which would be impossible.

In the case of non-compact topological manifold, Belitskii and Lyubich in [8] have proved the following theorem, that we present in a slightly less general setting than [8, Corollary 1.6], adapting their statements with respect to our purpose:

Theorem 4.4. *(From [8]) Assume that $\mathfrak{M}$ is locally compact and countable at infinity. If $f : \mathfrak{M} \to \mathfrak{M}$ is continuous and injective then the following statements are equivalent,*

(a) *There exists a continuous solution $\varphi : \mathfrak{M} \to \mathbb{C}$ of the Abel equation, $\varphi(f(x)) = \varphi(x) + 1$.*
(b) *For every continuous functions $p : \mathfrak{M} \to \mathbb{C}\backslash\{0\}$ and $\gamma : \mathfrak{M} \to \mathbb{C}$ there exits a continuous solution $\varphi : \mathfrak{M} \to \mathbb{C}$ of*
$$\varphi(f(x)) = p(x)\varphi(x) + \gamma(x). \tag{61}$$
(c) *Every compact subset of $\mathfrak{M}$ is wandering for f.*

In the above theorem, a compact set $K \subset \mathfrak{M}$, is qualified to be wandering if there exists an integer $\nu \geq 1$ such that
$$f^{\mathbf{n}}(K) \cap f^{\mathbf{m}}(K) = \emptyset \quad (n - m \geq \nu),$$
in particular such a dynamical system f is fixed-point free and periodic-point free, which is for instance consistent with dynamical restrictions imposed by any solution to the functional problem $\mathbf{P}_{\alpha}$ in the case $\lambda_{\mathbf{r}}(f) \geq 1$.[10]

This theorem provides however an incomplete answer to the problem $\mathbf{P}_{\alpha}$. For instance, Theorem 4.4 shows that for f satisfying condition (c) above, there exits a solution of the equation $\varphi(f(x)) = \lambda\varphi(x)$ with $|\lambda| \geq \alpha\lambda_{\mathbf{r}}(f)$, and therefore a solution of the generalized eigenvalue problem for $U_{\mathbf{f}}$; obtained by taking its module, $\Phi(\cdot) := |\varphi(\cdot)|$. The missing step for having a full solution of that problem is still the knowledge of the asymptotic behavior of Φ, which is also a difficult property to derive for solutions of cohomology equations; *e.g.* [7].

Remark 12. It is worth mentioning that in the framework developed in this article, the subset F plays a central role in the existence of solutions of functional problem of type $\mathbf{P}_{\alpha}$. The reader may consult for instance [10, §5.2], where it is shown that the Abel equation associated with a contracting mapping of the cut plane $\mathbb{R}^2\backslash(-\infty, 0]$, has a real-analytic solution, whereas the same map considered on the whole plane leads to an Abel equation without any continuous solution, since such a map possesses obviously a fixed point which is excluded by Theorem 4.4.

5. Concluding remarks.

From the previous section, whatever F and the approach retained, we can conclude that the main issue concerns therefore the asymptotic behavior of a possible eigenfunction of the generalized eigenvalue problem $\mathbf{P}_{\alpha}$. The regularity of the common eigenfunction is also an important aspect of the problem, in that respect, the continuity assumption on R and thus on Φ could be relaxed using Remark 1. Dynamical properties such as condition (c) of Theorem 4.4 might play also a role in the existence of such eigenfunctions. Note also, that since the closure of $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$ in the compact-open topology [25] is a closed cone with non

[10] Indeed, if there exists a generalized eigenfunction Φ of U_f and a periodic orbit of f of period p emanating from some x^*, then by repeating p-times the change of variable $x \leftarrow f(x)$ in $\Phi(f(x)) \geq \alpha\lambda_r(f)\Phi(x)$, we deduce necessarily that $(\alpha\lambda_r(f))^p \leq 1$ (since $\Phi > 0$), which imposes that f cannot possess such a periodic orbit in the case $\lambda_r(f) \geq 1$.

empty interior in $C^0(F, \mathbb{R})$ (cf. Remark 1-(a)), it would be interesting to study the spectral properties of $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$ within an approach of type Krein-Rutman theorem [46]. However, the task is more difficult in the present context than usually since F is assumed to be unbounded, σ-compact and locally compact, which implies that $C^0(F, \mathbb{R})$ has a Fréchet structure [2] (and not a Banach one), and in particular makes non straightforward an extension of the classical Krein-Rutman theorem in that context in order to analyze the existence of a principal eigenfunction in $\mathcal{E}_{\mathbf{F}}^{\mathbf{R}}$.

Finally, we have intentionally not considered important dynamical properties such as uniform hyperbolicity or its violation [29] that could lead to other spectral problem than $\mathbf{P}_{\alpha}$; this last one being presented here at a level of generality which lays the foundations for such enterprise. In that perspective, the cone condition (G_3) might be relaxed as suggested in Remark 1 (b), in order to bound the behavior of the generalized eigenfunction function $\Phi(x)$ by subadditive functions only for large values of x for instance, making thus the corresponding problem $\mathbf{P}_{\alpha}$ more flexible.

In summary, the purpose of the present work was to introduce, in a general context, a framework that makes apparent certain relations between the spectral theory of dynamical systems and the topological problem of conjugacy. Likewise, the idea of using some observable — here a common eigenfunction of the Koopman operators — to build a specific topology to deal with the conjugacy problem does not seem to be limited to the case of unbounded phase-space.

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